

# PRESSURE REGULARITY CRITERION FOR THE THREE DIMENSIONAL NAVIER-STOKES EQUATIONS IN INFINITE CHANNEL

CHONGSHENG CAO AND EDRISS S. TITI

ABSTRACT. In this paper we consider the three-dimensional Navier-Stokes equations in an infinite channel. We provide a sufficient condition, in terms of  $\partial_z p$ , where  $p$  is the pressure, for the global existence of the strong solutions to the three-dimensional Navier-Stokes equations.

AMS Subject Classifications: 35Q35, 65M70

Key words: Three-dimensional Navier-Stokes equations, Pressure regularity, Global regularity criterion for Navier-Stokes equations.

## 1. INTRODUCTION

The question of global regularity for the 3D Navier-Stokes equations is a major open problem in applied analysis. Over the years there has been an intensive work by many authors attacking this problem (see, e.g., [11], [12], [14], [22], [25], [27], [28], [29], [37], [40], [41], [42] and references therein). It is well known that the 2D Navier-Stokes equations have a unique weak and strong solutions which exist globally in time (cf., for example, [12], [40], [42]). In the 3D case, the weak solutions are known to exist globally in time. But, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the physical data of the initial-boundary value problem. Moreover this strong solution is known to be unique (cf., for example, [12], [37], [40]). The subtle difference between the 2D and 3D incompressible Navier-Stokes equations manifests itself in a clear way in the vorticity formulation of these equations. Let  $u$  denote the velocity field and  $\omega = \nabla \times u$  the vorticity. The equations that govern the evolution of the vorticity are given by

$$\begin{aligned} \frac{\partial \omega}{\partial t} - \nu \Delta \omega + (u \cdot \nabla) \omega - \omega \cdot \nabla u &= 0, \\ \nabla \cdot u &= 0. \end{aligned}$$

In this formulation the main obstacle for proving the global regularity is the vorticity stretching term  $\omega \cdot \nabla u$ . This term is identically equal to zero in the 2D case. Nonetheless, there are some results regarding the global regularity for the 3D Navier-Stokes equations under special symmetries and for which the vorticity stretching term is non-trivial. For example, the case of 3D axi-symmetric flows in domains of revolution (with a positive distance from the  $z$ -axis) [22] and [23] and helical flows [26]. It is worth mentioning, however, that the questions of the global regularity for 3D Euler equations (i.e., when the viscosity  $\nu = 0$ ) for axi-symmetric and helical flows with nontrivial vorticity stretching term are still open.

In addition, there are few known results regarding the global regularity for special types of initial data. For instance, for small  $H^1$  initial data (cf. e.g., [12], [40]). Fujita and Kato [15] proved the global well-posedness for small  $H^{1/2}$  initial data, and later Kato [20] proved the same result for small  $L^3$  initial data (see also [17]). An interesting global existence result for large, but very “oscillatory”, initial data was also proved in [4], [5], [6], [30] (see also references therein). The latter case can be regarded, roughly speaking, as small initial data in  $H^{1/2}$  and hence falls as a special case of results in [15]. The proofs of the above result relies heavily on the

---

*Date:* February 22, 2007.

viscosity mechanism. Roughly speaking, the viscosity acts very strongly in dissipating the solution starting from such oscillatory initial data so that in a very short time the solution becomes very small. As a result from that moment on a small initial data argument applies to establish the global existence. Here again, it is worth stressing that nothing is known about global regularity for the 3D Euler equations for this kind of oscillatory initial data. It is also worth mentioning the result of [34] about global existence of the strong solutions to the three-dimensional Navier–Stokes equations for large initial data in thin domains, where the upper bound on the size of the initial data depends inversely on the thickness of the domain.

Taking a physical point view, we define the dimensionless Reynolds number  $Re = \frac{\nu U_0}{\ell}$ , where  $U_0$  represents a typical magnitude of velocity (e.g. the size of the initial data) and  $\ell$  a typical length scale of the domain (e.g. its thickness). As a result one can view, roughly speaking, the above mentioned global regularity results to hold under the assumption of small enough Reynolds number.

The other direction concerning the question of global regularity for the 3D Navier–Stokes equations is to provide sufficient conditions for the global regularity. For example we refer the reader to the pioneer work of Prodi [33] and of Serrin [36] (see also the survey paper of Ladyzhenskaya [25] and references therein). Most recently, there has been some progress along these lines which states that a strong solution  $u$  exists on the time interval  $[0, T]$  as long as

$$u \in L^\alpha([0, T], L^\beta(\Omega)), \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1, \quad \text{for} \quad \beta > 3, \quad (1)$$

(see, for example, [2], [3], [17], [18], [20], [38], [39], and references therein). Moreover, there are some sufficient regularity conditions only on one component of the velocity field of the 3D NSE on the whole space  $\mathbb{R}^3$  or under periodic boundary conditions (cf. e.g., [19], [21], [32], [43]). In [7] we introduced a sufficient regularity condition on one component of the velocity field of the 3D Navier–Stokes equations under Dirichlet boundary conditions (see also, [31]).

The Navier–Stokes equations of viscous incompressible fluid in an infinite channel  $\mathbb{R}^2 \times (0, 1) \subset \mathbb{R}^3$  read:

$$\frac{\partial v}{\partial t} - \nu \Delta_h v - \nu v_{zz} + (v \cdot \nabla_h)v + wv_z + \nabla_h p = f, \quad (2)$$

$$\frac{\partial w}{\partial t} - \nu \Delta_h w - \nu w_{zz} + v \cdot \nabla_h w + ww_z + p_z = g, \quad (3)$$

$$\nabla_h \cdot v + w_z = 0, \quad (4)$$

where  $v = (v_1, v_2)$ , the horizontal velocity field components,  $w$ , the vertical velocity component, and  $p$ , the pressure are unknowns.  $(f, g)$ , the forcing term, and  $\nu > 0$ , the viscosity are given. We set  $\nabla_h = (\partial_x, \partial_y)$  to be the horizontal gradient operator and  $\Delta_h = \partial_x^2 + \partial_y^2$  is the horizontal Laplacian. We denote by:

$$\Gamma_u = \{(x, y, 1) \in \mathbb{R}^3\}, \quad (5)$$

$$\Gamma_b = \{(x, y, 0) \in \mathbb{R}^3\}, \quad (6)$$

the physical solid boundaries of the channel  $\mathbb{R}^2 \times (0, 1)$ . We equip the system (2)–(4) with the following no–normal flow and stress–free boundary conditions on the physical boundaries of the channel, namely,

$$\frac{\partial v}{\partial z} = 0, \quad w = 0; \quad \text{on } \Gamma_u \quad \text{and} \quad \Gamma_b. \quad (7)$$

Furthermore, we assume that  $(v, w)$  is periodic with period 1 in both horizontal directions. That is,

$$v(x+1, y, z) = v(x, y+1, z) = v(x, y, z), \quad w(x+1, y, z) = w(x, y+1, z) = w(x, y, z). \quad (8)$$

Because of the horizontal periodic boundary conditions we consider here

$$\Omega = (0, 1)^3$$

as our basic domain of the flow, and we denote by

$$M = (0, 1)^2.$$

In addition, we supply the system with the initial condition:

$$v(x, y, z, 0) = v_0(x, y, z), \quad (9)$$

$$w(x, y, z, 0) = w_0(x, y, z). \quad (10)$$

Let us denote by  $L^q(\Omega)$  and  $H^m(\Omega)$  the usual  $L^q$ -Lebesgue and Sobolev spaces, respectively (cf. [1]). We denote by

$$\|\phi\|_q = \left( \int_{\Omega} |\phi|^q \, dx dy dz \right)^{\frac{1}{q}}, \quad \text{for every } \phi \in L^q(\Omega). \quad (11)$$

Let

$$\begin{aligned} \mathcal{V}_1 &= \left\{ \phi \in (C^\infty(\mathbb{R}^2 \times [0, 1]))^2 : \frac{\partial \phi}{\partial z} \Big|_{z=0} = 0; \frac{\partial \phi}{\partial z} \Big|_{z=1} = 0; \phi(x+1, y, z) = \phi(x, y+1, z) = \phi(x, y, z) \right\}, \\ \mathcal{V}_2 &= \left\{ \psi \in C^\infty(\mathbb{R}^2 \times [0, 1]) : \psi|_{z=0} = 0; \psi|_{z=1} = 0; \psi(x+1, y, z) = \psi(x, y+1, z) = \psi(x, y, z) \right\}, \\ \mathcal{V} &= \{(\phi, \psi) : \phi \in \mathcal{V}_1, \psi \in \mathcal{V}_2, \nabla_h \cdot \phi + \psi_z = 0\}. \end{aligned}$$

We denote by  $H$  and  $V$  be the closure spaces of  $\mathcal{V}$  in  $L^2(\Omega)$  under the  $L^2$ -topology, and in  $H^1(\Omega)$  under the  $H^1$ -topology, respectively. We say  $(v, w)$  is a Leray-Hopf weak solution to the system (2)–(10) if  $(v, w)$  satisfies

- (1)  $(v, w) \in C([0, T], H) \cap L^2([0, T], V)$ , and  $(\partial_t v, \partial_t w) \in L^1([0, T], V')$ , where  $V'$  is the dual space of  $V$ ;
- (2) the weak formulation:

$$\begin{aligned} & \int_{\Omega} (v \cdot \phi + w \psi) \, dx dy dz - \int_{\Omega} (v(t_0) \cdot \phi(t_0) + w(t_0) \psi(t_0)) \, dx dy dz \\ &= \int_{t_0}^t \int_{\Omega} (v, w) \cdot ((\phi_t, \psi_t) + \nu(\Delta_h \phi + \phi_{zz}, \Delta_h \psi + \psi_{zz})) \, dx dy dz \, ds \\ &+ \int_{t_0}^t \int_{\Omega} [(v \cdot \nabla_h)(\phi, \psi) \cdot (v, w) + w(\phi_z, \psi_z) \cdot (v, w)] \, dx dy dz + \int_{t_0}^t \int_{\Omega} [(f, g) \cdot (v, w)] \, dx dy dz, \end{aligned}$$

for every  $(\phi, \psi) \in \mathcal{V}$ , and almost every  $t, t_0 \in [0, T]$ ;

- (3) the energy inequality:

$$\frac{1}{2} \frac{d(\|v\|_2^2 + \|w\|_2^2)}{dt} + \nu [\|\nabla_h v\|_2^2 + \|\nabla_h w\|_2^2 + \|v_z\|_2^2 + \|w_z\|_2^2] \leq \int_{\Omega} (f, g) \cdot (v, w) \, dx dy dz. \quad (12)$$

Moreover, a weak solution is called strong solution of (2)–(10) on  $[0, T]$  if, in addition, it satisfies

$$(v, w) \in C([0, T], V) \cap L^2([0, T], H^2(\Omega)).$$

*Remark 1.* Notice that one can extend  $v$  and  $w$  to be periodic, vertically, by, first, setting

$$\begin{aligned} v(x, y, z, t) &= v(x, y, -z, t), \quad z \in (-1, 0) \\ w(x, y, z, t) &= -w(x, y, -z, t), \quad z \in (-1, 0), \end{aligned}$$

and then, setting

$$v(x, y, z+2, t) = v(x, y, z, t) \quad w(x, y, z+2, t) = w(x, y, z, t) \quad \forall z \in \mathbb{R}.$$

Similarly,  $p$ ,  $f$ , and  $v_0$  can be extended to be even periodic functions in the  $z$ -vertical with period 2, and  $g$  and  $w_0$  are odd periodic functions in the  $z$ -vertical with period 2. In this way, the system (2)–(10) can be treated as the 3D Navier-Stokes equations with periodic boundary condition. In other words the system (2)–(10) is a special case of the 3D Navier-Stokes equations on  $\mathbb{R}^3$  with periodic boundary condition. Then, by standard procedure for the 3D Navier-Stokes equations with periodic boundary condition (see, e.g., [12], [14], [25], [34],

[37], [40], [41]) one can show that there exists, global in time, a Leray–Hopf weak solution to the system (2)–(10) if  $(v_0, w_0) \in H$ . Furthermore, one can show the short time existence of the strong solution if  $(v_0, w_0) \in V$ .

In this paper, we provide sufficient conditions on the pressure which guarantee the global existence of the strong solution to the 3D Navier–Stokes equations in infinite channel subject to the boundary conditions (7)–(8). Several authors (see, for example, [3], [10], [13], [35], [44], [45]) have studied the question of the global regularity of the 3D Navier–Stokes equation by providing sufficient conditions on the pressure. Specifically, the authors of [35] have shown that if  $|u|^2 + 2p$  is bounded from below or from above then the weak solution is strong solution. The authors of [10] have shown that if  $p$  satisfies

$$p \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\beta} < 2, \quad \text{for} \quad \beta > \frac{3}{2}, \quad (13)$$

then the weak solution is strong solution. This result has been improved in [3] by assuming

$$p \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2, \quad \text{for} \quad \beta > \frac{3}{2}. \quad (14)$$

In [45] the author has established the global regularity by assuming that

$$\nabla_h p, p_z \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha} + \frac{3}{\beta} \leq 3, \quad \text{for} \quad \alpha > \frac{2}{3}, \quad \beta > 1. \quad (15)$$

Here, we will show the existence of the strong solutions of the system (2)–(10) on interval  $[0, T]$  provided the vertical derivative of the pressure satisfies

$$p_z \in L^\alpha([0, T]; L^\beta(\Omega)) \quad \text{with} \quad \alpha > 3, \quad \text{and} \quad \beta > 2. \quad (16)$$

Let us observe that the quantity that appears in the Navier–Stokes equations is  $(\nabla_h p, p_z)$  and not  $p$  itself. Therefore, the conditions from [45] and our conditions seem to be natural. Furthermore, it is worth mentioning that the conditions (16) seem not to be comparable in any direct way with (13), (14) and (15). Nevertheless, our tools are totally different from those in [3], [10] and [45]. We prove our result by using the methods we established in [9] in which we proved the global well-posedness of the 3D viscous primitive equations. Finally, notice that the conditions (1) and (14) are scaling invariant when the equation is considered on whole  $\mathbb{R}^3$ . Namely,

$$\left( \int_0^T \|u_\lambda\|_\beta^\alpha dt \right)^{\frac{1}{\alpha}} = \left( \int_0^T \|u\|_\beta^\alpha dt \right)^{\frac{1}{\alpha}}, \quad \text{when} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1,$$

and

$$\left( \int_0^T \|p_\lambda\|_\beta^\alpha dt \right)^{\frac{1}{\alpha}} = \left( \int_0^T \|p\|_\beta^\alpha dt \right)^{\frac{1}{\alpha}}, \quad \text{when} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2,$$

where  $u_\lambda = \lambda u(\lambda x, \lambda y, \lambda z, \lambda^2 t)$  and  $p_\lambda = \lambda^2 u(\lambda x, \lambda y, \lambda z, \lambda^2 t)$  which is also a solution to the Navier–Stokes equation if  $(u, p)$  is a solution to the Navier–Stokes equation in whole  $\mathbb{R}^3$ . However, our condition is not scaling invariant. The reason may be because either our results are not optimal, or because our condition involves only one partial derivative of the pressure term and our result is limited to channel flows and not in the whole space  $\mathbb{R}^3$ , where the scaling argument is applied.

The plan of this paper is as follows. In section 2, we reformulate the system (2)–(10), introduce our notations, and recall some well-known useful inequalities. Section 3 is the main section, and it is devoted for the regularity of solutions.

## 2. PRELIMINARIES AND FUNCTIONAL SETTING

In this section we introduce a new equivalent formulation of (2)–(10). Following the ideas introduced in [9] we integrate the equation (4) in the  $z$  direction and by (7) we obtain

$$w(x, y, z, t) = - \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi, \quad (17)$$

and

$$\int_0^1 \nabla_h \cdot v(x, y, z, t) dz = \nabla_h \cdot \int_0^1 v(x, y, z, t) dz = 0. \quad (18)$$

For every function  $\theta(x, y, z)$ , we denote by

$$\bar{\theta}(x, y) = \int_0^1 \theta(x, y, z) dz. \quad (19)$$

and

$$\tilde{\theta} = \theta - \bar{\theta}.$$

Following the geophysical terminology we denote by

$$\bar{v}(x, y) = \frac{1}{h} \int_{-h}^0 v(x, y, \xi) d\xi, \quad \text{in } M, \quad (20)$$

the barotropic mode. We will also denote by

$$\tilde{v} = v - \bar{v}, \quad (21)$$

the baroclinic mode, that is the fluctuation about the barotropic mode. Notice that

$$\bar{\tilde{\theta}} = 0. \quad (22)$$

By substituting (17) into (2), we reach

$$\frac{\partial v}{\partial t} - \nu \Delta_h v - \nu v_{zz} + (v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z} + \nabla_h p = f, \quad (23)$$

and

$$\nabla_h \cdot \bar{v} = 0. \quad (24)$$

Furthermore, by taking the average of equations (23) in the  $z$  direction, over the interval  $(0, 1)$  and using the boundary conditions (7), we reach

$$\frac{\partial \bar{v}}{\partial t} - \nu \Delta_h \bar{v} + \overline{(v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z}} + \nabla_h \bar{p} = \bar{f}. \quad (25)$$

As a result of (18), (22), and integration by parts and using the boundary conditions (7), the nonlinear term in (25) gives

$$\overline{(v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z}} = (\bar{v} \cdot \nabla_h) \bar{v} + \overline{[(\tilde{v} \cdot \nabla_h) \tilde{v} + (\nabla_h \cdot \tilde{v}) \tilde{v}]}. \quad (26)$$

By subtracting (25) from (23) and using (26) we obtain

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} - \nu \Delta_h \tilde{v} - \nu \tilde{v}_{zz} + (\tilde{v} \cdot \nabla_h) \tilde{v} - \left( \int_0^z \nabla_h \cdot \tilde{v}(x, y, \xi, t) d\xi \right) \frac{\partial \tilde{v}}{\partial z} \\ + (\tilde{v} \cdot \nabla_h) \bar{v} + (\bar{v} \cdot \nabla_h) \tilde{v} - \overline{[(\tilde{v} \cdot \nabla_h) \tilde{v} + (\nabla_h \cdot \tilde{v}) \tilde{v}]} + \nabla_h \tilde{p} = \tilde{f}. \end{aligned} \quad (27)$$

In addition,  $\tilde{v}$  satisfies the boundary conditions:

$$\tilde{v}(x+1, y, z) = \tilde{v}(x, y+1, z) = v(x, y, z), \quad (28)$$

$$\left. \frac{\partial \tilde{v}}{\partial z} \right|_{z=0} = 0, \quad \left. \frac{\partial \tilde{v}}{\partial z} \right|_{z=1} = 0. \quad (29)$$

For convenience, we recall the following Gagliardo–Nirenberg, Sobolev and Ladyzhenskaya inequalities in  $\mathbb{R}^2$  (see, e.g., [1], [12], [16], [24])

$$\|\phi\|_{L^\alpha(M)} \leq C_\alpha \|\phi\|_{L^2(M)}^{2/\alpha} \|\phi\|_{H^1(M)}^{\frac{\alpha-2}{\alpha}}, \quad (30)$$

for every  $\phi \in H^1(M)$ ,  $2 \leq \alpha < \infty$ , and the following Gagliardo–Nirenberg, Sobolev and Ladyzhenskaya inequalities in  $\mathbb{R}^3$  (see, e.g., [1], [12], [16], [24])

$$\|\psi\|_{L^\alpha(\Omega)} \leq C_\alpha \|\psi\|_{L^2(\Omega)}^{\frac{6-\alpha}{2\alpha}} \|\psi\|_{H^1(\Omega)}^{\frac{3(\alpha-2)}{2\alpha}}, \quad (31)$$

for every  $\psi \in H^1(\Omega)$ ,  $2 \leq \alpha \leq 6$ . Here  $C_\alpha$  denote constants which are scale invariant. Also, by (30) we get

$$\begin{aligned} \|\phi\|_{L^\beta(M)} &= \| |\phi|^{\alpha/2} \|_{L^{2\beta/\alpha}(M)}^{2/\alpha} \leq C \| |\phi|^{\alpha/2} \|_{L^2(M)}^{2/\beta} \| |\phi|^{\alpha/2} \|_{H^1(M)}^{\frac{2(\beta-\alpha)}{\alpha\beta}} \\ &\leq C \|\phi\|_{L^\alpha(M)}^{\alpha/\beta} \left( \int_M |\phi|^{\alpha-2} |\nabla_h \phi|^2 dx dy \right)^{\frac{(\beta-\alpha)}{\alpha\beta}} + \|\phi\|_{L^\alpha(M)}, \end{aligned} \quad (32)$$

for every  $\phi \in H^1(M)$ , and  $\beta > \alpha$ . Also, we recall the integral version of Minkowsky inequality for the  $L^\beta$  spaces,  $\beta \geq 1$ . Let  $\Omega_1 \subset \mathbb{R}^{m_1}$  and  $\Omega_2 \subset \mathbb{R}^{m_2}$  be two measurable sets, where  $m_1$  and  $m_2$  are two positive integers. Suppose that  $f(\xi, \eta)$  is measurable over  $\Omega_1 \times \Omega_2$ . Then,

$$\left[ \int_{\Omega_1} \left( \int_{\Omega_2} |f(\xi, \eta)| d\eta \right)^\beta d\xi \right]^{1/\beta} \leq \int_{\Omega_2} \left( \int_{\Omega_1} |f(\xi, \eta)|^\beta d\xi \right)^{1/\beta} d\eta. \quad (33)$$

### 3. EXISTENCE OF THE STRONG SOLUTION

In this section we will prove the global existence of the strong solution to the system (2)–(10) under the assumption (16) on  $p_z$ .

**Theorem 1.** *Suppose that  $f, g \in H^1(\Omega)$ . For every  $(v_0, w_0) \in V$ , and if  $p_z \in L^\alpha([0, T], L^{2q})$  with  $T > 0$ ,  $\alpha > 3$  and  $q > 1$ , then there is a unique strong solution  $((v, w), p)$  of the system (2)–(4) on  $[0, T]$ .*

*Proof.* In Remark 1 we described an argument that guarantees the existence of a Leray–Hopf weak solution and short time existence of the strong solutions. Suppose that  $((v, w), p)$  is the strong solution with initial value  $(v_0, w_0) \in V$  such that  $(v, w) \in C([0, T^*), V) \cap L^2([0, T^*), H^2(\Omega))$ , where  $[0, T^*)$ , for  $T^* \leq T$ , is the maximal interval of existence. If  $T^* = T$ , then there is nothing to prove. Next, we would like to show that certain norms of this strong solution remain finite for all the time, up to  $T^*$ , provided the condition (16) is valid. In this way we show that  $T^*$  is, at least, equal to  $T$ . Namely, the strong solution  $((v, w), p)$  exists on  $[0, T]$ . By the energy inequality (12) we have (see, for example, [12], [37], [40] for details)

$$\|v(t)\|_2^2 + \|w(t)\|_2^2 \leq K_{11}, \quad (34)$$

and

$$\nu \int_0^t [\|\nabla_h v(s)\|_2^2 + \|\nabla_h w(s)\|_2^2 + \|v_z(s)\|_2^2 + \|w_z(s)\|_2^2] ds \leq K_{12}(t), \quad (35)$$

where

$$K_{11} = \frac{\|f\|_2^2 + \|g\|_2^2}{\nu^2 \lambda_1^2} + \|v_0\|_2^2 + \|w_0\|_2^2. \quad (36)$$

$$K_{12}(t) = \frac{(\|f\|_2^2 + \|g\|_2^2) t}{\nu \lambda_1} + \|v_0\|_2^2 + \|w_0\|_2^2. \quad (37)$$

3.1.  $\|\tilde{v}\|_r, 3 < r < 4$  **estimates.** Taking the inner product of the equation (27) with  $|\tilde{v}|^{r-2}\tilde{v}$  in  $L^2(\Omega)$ , we get

$$\begin{aligned} & \frac{1}{r} \frac{d\|\tilde{v}\|_r^r}{dt} + \nu \int_{\Omega} \left( |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + (r-2) |\nabla_h |\tilde{v}||^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2} + (r-2) |\partial_z |\tilde{v}||^2 |\tilde{v}|^{r-2} \right) dx dy dz \\ &= - \int_{\Omega} \left( (\tilde{v} \cdot \nabla_h) \tilde{v} - \left( \int_{-h}^z \nabla_h \cdot \tilde{v}(x, y, \xi, t) d\xi \right) \frac{\partial \tilde{v}}{\partial z} + (\tilde{v} \cdot \nabla_h) \bar{v} + (\bar{v} \cdot \nabla_h) \tilde{v} \right. \\ & \quad \left. - \overline{(\tilde{v} \cdot \nabla_h) \tilde{v}} + \overline{(\nabla_h \cdot \tilde{v}) \tilde{v}} + \nabla_h \tilde{p} - \tilde{f} \right) \cdot |\tilde{v}|^{r-2} \tilde{v} dx dy dz. \end{aligned}$$

By integration by parts and using the boundary conditions (28) and (29) we get

$$- \int_{\Omega} \left( (\tilde{v} \cdot \nabla_h) \tilde{v} - \left( \int_0^z \nabla_h \cdot \tilde{v}(x, y, \xi, t) d\xi \right) \frac{\partial \tilde{v}}{\partial z} \right) \cdot |\tilde{v}|^{r-2} \tilde{v} dx dy dz = 0. \quad (38)$$

Moreover, (24) and the boundary condition (29) give us

$$\int_{\Omega} (\bar{v} \cdot \nabla_h) \tilde{v} \cdot |\tilde{v}|^{r-2} \tilde{v} dx dy dz = 0. \quad (39)$$

Thus, from (38)–(39) we have

$$\begin{aligned} & \frac{1}{r} \frac{d\|\tilde{v}\|_r^r}{dt} + \nu \int_{\Omega} \left( |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + (r-2) |\nabla_h |\tilde{v}||^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2} + (r-2) |\partial_z |\tilde{v}||^2 |\tilde{v}|^{r-2} \right) dx dy dz \\ &= - \int_{\Omega} \left( (\tilde{v} \cdot \nabla_h) \bar{v} - \overline{(\tilde{v} \cdot \nabla_h) \tilde{v}} + \overline{(\nabla_h \cdot \tilde{v}) \tilde{v}} + \nabla_h \tilde{p} - \tilde{f} \right) \cdot |\tilde{v}|^{r-2} \tilde{v} dx dy dz. \end{aligned}$$

Notice that by integration by parts and using boundary conditions (28) and (29) we have

$$\begin{aligned} & - \int_{\Omega} \left[ (\tilde{v} \cdot \nabla_h) \bar{v} - \overline{(\tilde{v} \cdot \nabla_h) \tilde{v}} + \overline{(\nabla_h \cdot \tilde{v}) \tilde{v}} + \nabla_h \tilde{p} \right] \cdot |\tilde{v}|^{r-2} \tilde{v} dx dy dz \\ &= \int_{\Omega} \left[ (\nabla_h \cdot \tilde{v}) \bar{v} \cdot |\tilde{v}|^{r-2} \tilde{v} + (\tilde{v} \cdot \nabla_h) (|\tilde{v}|^{r-2} \tilde{v}) \cdot \bar{v} - \sum_{k,j=1}^2 \left[ \overline{v^k v^j} \partial_{x_k} (|\tilde{v}|^{r-2} \tilde{v}^j) \right] + \tilde{p} (\nabla_h \cdot (|\tilde{v}|^{r-2} \tilde{v})) \right] dx dy dz. \end{aligned}$$

Observe that since  $\tilde{p} = 0$ , we have the Poincaré inequality

$$|\tilde{p}| \leq \int_{-h}^0 |p_z| dz. \quad (40)$$

Therefore, from all the above and by Cauchy–Schwarz and Hölder inequalities we obtain

$$\begin{aligned}
& \frac{1}{r} \frac{d\|\tilde{v}\|_r^r}{dt} + \nu \int_{\Omega} \left( |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + (r-2) |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2} + (r-2) |\partial_z \tilde{v}|^2 |\tilde{v}|^{r-2} \right) dx dy dz \\
& \leq C \int_M \left[ |\tilde{v}| \int_0^1 |\nabla_h \tilde{v}| |\tilde{v}|^{r-1} dz \right] dx dy \\
& \quad + C \int_M \left[ \int_0^1 |\tilde{v}|^2 dz \int_0^1 |\nabla_h \tilde{v}| |\tilde{v}|^{r-2} dz \right] dx dy \\
& \quad + C \int_M \left[ \int_0^1 |p_z| dz \int_0^1 |\nabla_h \tilde{v}| |\tilde{v}|^{r-2} dz \right] dx dy + \|\tilde{f}\|_r \|\tilde{v}\|_r^{r-1} \\
& \leq C \int_M \left[ |\tilde{v}| \left( \int_0^1 |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dz \right)^{1/2} \left( \int_0^1 |\tilde{v}|^r dz \right)^{1/2} \right] dx dy \\
& \quad + C \int_M \left[ \int_0^1 |\tilde{v}|^2 dz \left( \int_0^1 |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dz \right)^{1/2} \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{1/2} \right] dx dy \\
& \quad + C \int_M \left[ \int_0^1 |p_z| dz \left( \int_0^1 |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dz \right)^{1/2} \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{1/2} \right] dx dy + \|\tilde{f}\|_r \|\tilde{v}\|_r^{r-1} \\
& \leq C \|\tilde{v}\|_{L^4(M)} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dx dy dz \right)^{1/2} \left( \int_M \left( \int_0^1 |\tilde{v}|^r dz \right)^2 dx dy \right)^{1/4} \\
& \quad + C \left( \int_M \left( \int_0^1 |\tilde{v}|^2 dz \right)^{\frac{r+2}{2}} dx dy \right)^{\frac{2}{r+2}} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dx dy dz \right)^{\frac{1}{2}} \left( \int_M \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{\frac{r+2}{r-2}} dx dy \right)^{\frac{r-2}{2(r+2)}} \\
& \quad + C \|\overline{p_z}\|_{L^{2q}(M)} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dx dy dz \right)^{1/2} \left( \int_M \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{q'} dx dy \right)^{1/2q'},
\end{aligned}$$

where  $1/q + 1/q' = 1$ . By using Minkowsky inequality (33), we get

$$\left( \int_M \left( \int_0^1 |\tilde{v}|^r dz \right)^2 dx dy \right)^{1/2} \leq C \int_0^1 \left( \int_M |\tilde{v}|^{2r} dx dy \right)^{1/2} dz.$$

By virtue of (32), we have

$$\int_M |\tilde{v}|^{2r} dx dy \leq C_0 \int_M |\tilde{v}|^r dx dy \int_M |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dx dy + \left( \int_M |\tilde{v}|^r dx dy \right)^2.$$

Thus, by Cauchy–Schwarz inequality we obtain

$$\left( \int_M \left( \int_0^1 |\tilde{v}|^r dz \right)^2 dx dy \right)^{1/2} \leq C \|\tilde{v}\|_r^{r/2} \left( \int_{\Omega} |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dx dy dz \right)^{1/2} + \|\tilde{v}\|_r^r. \quad (41)$$



Similarly, by (33) and (32) and (30), we also obtain

$$\begin{aligned}
& \left( \int_M \left( \int_0^1 |\tilde{v}|^2 dz \right)^{(r+2)/2} dxdy \right)^{2/(r+2)} \leq C \int_0^1 \left( \int_M |\tilde{v}|^{2+r} dxdy \right)^{2/(r+2)} dz \\
& \leq C \int_0^1 \left[ \|\tilde{v}\|_{L^r(M)}^{\frac{2r}{r+2}} \left( \int_M |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdy \right)^{\frac{4}{r(r+2)}} + \|\tilde{v}\|_{L^r(M)}^2 \right] dz \\
& \leq C \int_0^1 \left[ \|\tilde{v}\|_{L^2(M)}^{\frac{2}{r}} \|\nabla_h \tilde{v}\|_{L^2(M)}^{\frac{r-2}{r}} \|\tilde{v}\|_{L^r(M)}^{\frac{r-2}{r+2}} \left( \int_M |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdy \right)^{\frac{4}{r(r+2)}} + \|\tilde{v}\|_{L^r}^2 \right] dz \\
& \leq C \|\tilde{v}\|_2^{\frac{2}{r}} \|\nabla_h \tilde{v}\|_2^{\frac{r-2}{r}} \|\tilde{v}\|_r^{\frac{r-2}{r+2}} \left( \int_\Omega |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdydz \right)^{\frac{4}{r(r+2)}} + \|\tilde{v}\|_r^2,
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
& \left( \int_M \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{(r+2)/(r-2)} dxdy \right)^{\frac{r-2}{r+2}} \leq C \int_0^1 \left( \int_M |\tilde{v}|^{2+r} dxdy \right)^{(r-2)/(r+2)} dz \\
& \leq C \int_0^1 \left[ \|\tilde{v}\|_{L^r(M)}^{\frac{r(r-2)}{r+2}} \left( \int_M |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdy \right)^{\frac{2(r-2)}{r(r+2)}} + \|\tilde{v}\|_{L^r(M)}^{r-2} \right] dz \\
& \leq C \|\tilde{v}\|_r^{\frac{r(r-2)}{r+2}} \left( \int_\Omega |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdydz \right)^{\frac{2(r-2)}{r(r+2)}} + \|\tilde{v}\|_r^{r-2},
\end{aligned} \tag{43}$$

Thanks to (30) and (33), we conclude

$$\begin{aligned}
& \left( \int_M \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{\frac{r-2}{r}} dxdy \right)^{\frac{r-2}{r}} \leq C \int_0^1 \left( \int_M |\tilde{v}|^r dxdy \right)^{\frac{r-2}{r}} dz \\
& \leq C \int_0^1 \left[ \|\tilde{v}\|_{L^2(M)}^{\frac{2(r-2)}{r}} \|\nabla_h \tilde{v}\|_{L^2(M)}^{\frac{(r-2)^2}{r}} + \|\tilde{v}\|_{L^2(M)}^{r-2} \right] dz \leq C \|\tilde{v}\|_2^{\frac{2(r-2)}{r}} \|\nabla_h \tilde{v}\|_2^{\frac{(r-2)^2}{r}} + \|\tilde{v}\|_{L^2(M)}^{r-2}.
\end{aligned} \tag{44}$$

Moreover,

$$\begin{aligned}
& \left( \int_M \left( \int_0^1 |\tilde{v}|^{r-2} dz \right)^{q'} dxdy \right)^{1/q'} \leq C \int_0^1 \left( \int_M |\tilde{v}|^{(r-2)q'} dxdy \right)^{1/q'} dz \\
& \leq C \int_0^1 \left[ \|\tilde{v}\|_{L^r(M)}^{r/q'} \left( \int_M |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdy \right)^{\frac{(r-2)q'-r}{rq'}} + \|\tilde{v}\|_{L^r(M)}^{(r-2)} \right] dz \\
& \leq C \|\tilde{v}\|_r^{r/q'} \left( \int_\Omega |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dxdydz \right)^{\frac{(r-2)q'-r}{rq'}} + \|\tilde{v}\|_r^{(r-2)}.
\end{aligned} \tag{45}$$

Therefore, (41)–(45) and (30) give

$$\begin{aligned}
& \frac{d\|\tilde{v}\|_r^r}{dt} + \nu \int_{\Omega} \left( |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + |\nabla_h |\tilde{v}|^2|^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2} + |\partial_z |\tilde{v}|^2|^2 |\tilde{v}|^{r-2} \right) dx dy dz \\
& \leq C \|\bar{v}\|_2^{1/2} \|\nabla_h \bar{v}\|_2^{1/2} \|\tilde{v}\|_r^{r/2} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^4 dx dy dz \right)^{3/4} + C \|\bar{v}\|_2^{1/2} \|\nabla_h \bar{v}\|_2^{1/2} \|\tilde{v}\|_r^r + \|f\|_r \|\tilde{v}\|_r^{r-1} + C \|\tilde{v}\|_r^r \\
& \quad + C \|\tilde{v}\|_2^{\frac{2}{r}} \|\nabla_h \tilde{v}\|_2^{\frac{r-2}{r}} \|\tilde{v}\|_r^{\frac{r-2}{2}} \left( \int_{\Omega} |\tilde{v}|^{r-2} |\nabla_h \tilde{v}|^2 dx dy dz \right)^{\frac{r+2}{2r}} \\
& \quad + C \|p_z\|_{2q} \|\tilde{v}\|_r^{\frac{r}{2q}} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dx dy dz \right)^{1 - \frac{r+2q'}{2rq'}} + C \|p_z\|_{2q} \|\tilde{v}\|_r^{(r-2)/2} \left( \int_{\Omega} |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} dx dy dz \right)^{1/2}.
\end{aligned}$$

By Young's and Cauchy–Schwarz inequalities we have

$$\begin{aligned}
& \frac{d\|\tilde{v}\|_r^r}{dt} + \nu \int_{\Omega} \left( |\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + |\nabla_h |\tilde{v}|^2|^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2} + |\partial_z |\tilde{v}|^2|^2 |\tilde{v}|^{r-2} \right) dx dy dz \\
& \leq C \left[ 1 + \|\bar{v}\|_2^2 \|\nabla_h \bar{v}\|_2^2 + \|\tilde{v}\|_2^{\frac{4}{r-2}} \|\nabla_h \tilde{v}\|_2^2 \right] \|\tilde{v}\|_r^r + C \|p_z\|_{2q}^r + C \|f\|_r^r.
\end{aligned}$$

Thanks to Gronwall inequality, we get

$$\|\tilde{v}(t)\|_r^r + \nu \int_0^t \int_{\Omega} [|\nabla_h \tilde{v}|^2 |\tilde{v}|^{r-2} + |\tilde{v}_z|^2 |\tilde{v}|^{r-2}] dx dy dz \leq K_R, \quad (46)$$

where

$$K_R = e^{CT + K_{11}K_{12}(T) + K_{11}^{\frac{2}{r-2}}K_{12}(T)} \left[ 1 + \|v_0\|_{H^1(\Omega)}^6 + \int_0^T \|p_z(s)\|_{2q}^r ds + \|f\|_r^r T \right]. \quad (47)$$

It is worth mentioning that the above estimate is also valid for  $2 \leq r < 4$ . However, one need  $r > 3$  in order to get the following  $H^1$  estimate.

**3.2.  $H^1$  estimates.** Before we show the global  $H^1$  bound, let us prove the following Lemma.

**Lemma 2.** *Let  $\phi \in H^1(\Omega)$ ,  $\psi \in L^2(\Omega)$ , and  $v, r$  be as in Theorem 1. Then,*

$$\int_{\Omega} |v| |\phi| |\psi| dx dy dz \leq \epsilon (\|\nabla \phi\|_2^2 + \|\phi_z\|_2^2 + \|\psi\|_2^2) \quad (48)$$

$$+ C_{\epsilon} \left[ \|\tilde{v}\|_r^{\frac{2r}{r-3}} + \|\tilde{v}\|_r^2 + (1 + \|\bar{v}\|_2^2) (\|\bar{v}\|_2^2 + \|\nabla_h \bar{v}\|_2^2) \right] \|\phi\|_2^2 \quad (49)$$

for every  $\epsilon > 0$ .

*Proof.* Notice that

$$\int_{\Omega} |v| |\phi| |\psi| dx dy dz \leq \int_{\Omega} (|\tilde{v}| + |\bar{v}|) |\phi| |\psi| dx dy dz.$$

By Hölder inequality and (30), we obtain

$$\begin{aligned}
& \int_{\Omega} |\tilde{v}| |\phi| |\psi| dx dy dz \leq \|\tilde{v}\|_r \|\phi\|_{\frac{2r}{r-2}} \|\psi\|_2 \\
& \leq C \|\tilde{v}\|_r \|\phi\|_2^{\frac{r-3}{r}} (\|\nabla \phi\|_2 + \|\phi_z\|_2)^{\frac{3}{r}} \|\psi\|_2 + \|\tilde{v}\|_r \|\phi\|_2 \|\psi\|_2.
\end{aligned}$$

By Young's inequality, we reach

$$\int_{\Omega} |\tilde{v}| |\phi| |\psi| dx dy dz \leq \frac{\epsilon}{2} (\|\nabla \phi\|_2^2 + \|\phi_z\|_2^2 + \|\psi\|_2^2) + C_{\epsilon} \left( \|\tilde{v}\|_r^{\frac{2r}{r-3}} + \|\tilde{v}\|_r^2 \right) \|\phi\|_2^2.$$

On the other hand, by applying the same method for proving Proposition 2.2 in [8], we get

$$\int_{\Omega} |\bar{v}| |\phi| |\psi| \, dx dy dz \leq C \left( \|\bar{v}\|_2^{1/2} \|\nabla_h \bar{v}\|_2^{1/2} + \|\bar{v}\|_2 \right) \left( \|\phi\|_2^{1/2} \|\nabla_h \phi\|_2 + \|\phi\|_2 \right) \|\psi\|_2.$$

Again, by Young's inequality, we obtain

$$\int_{\Omega} |v| |\phi| |\psi| \, dx dy dz \leq \frac{\epsilon}{2} (\|\nabla_h \phi\|_2^2 + \|\psi\|_2^2) + C_{\epsilon} (1 + \|\bar{v}\|_2^2) (\|\bar{v}\|_2^2 + \|\nabla_h \bar{v}\|_2^2) \|\phi\|_2^2.$$

Therefore, (49) holds.  $\square$

Next, let show the  $H^1$  norm of the strong solution  $(v, w)$  is bounded. Taking the inner product of the equation (2) with  $-\Delta_h v - v_{zz}$  and the equation (3) with  $-\Delta_h w - w_{zz}$  in  $L^2$ , and using the fact that the Stokes operator is same as the Laplacian operator under periodic boundary conditions, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d (\|\nabla_h v\|_2^2 + \|v_z\|_2^2 + \|\nabla_h w\|_2^2 + \|w_z\|_2^2)}{dt} + \nu (\|\Delta_h v\|_2^2 + 2\|\nabla_h v_z\|_2^2 + \|v_{zz}\|_2^2 + \|\Delta_h w\|_2^2 + 2\|\nabla_h w_z\|_2^2 + \|w_{zz}\|_2^2) \\ &= \int_{\Omega} [(v \cdot \nabla_h) v + w v_z - f] \cdot (\Delta_h v + v_{zz}) \, dx dy dz + \int_{\Omega} [v \cdot \nabla_h w + w w_z - g] (\Delta_h w + w_{zz}) \, dx dy dz. \end{aligned}$$

By integration by parts and using the boundary conditions (7) and (8), we obtain

$$\begin{aligned} & \int_{\Omega} w v_z \cdot (\Delta_h v + v_{zz}) \, dx dy dz \\ &= - \int_{\Omega} ((\nabla_h w \cdot \nabla_h v) \cdot v_z + w_z |v_z|^2) \, dx dy dz \\ &= \int_{\Omega} (\nabla_h w_z \cdot \nabla_h v + \nabla_h w \cdot \nabla_h v_z + w_{zz} v_z + w_z v_{zz}) \cdot v \, dx dy dz, \end{aligned}$$

and also

$$\begin{aligned} & \int_{\Omega} w w_z (\Delta_h w + w_{zz}) \, dx dy dz \\ &= - \int_{\Omega} \left( |\nabla_h w|^2 w_z + w \nabla_h w_z \cdot \nabla_h w + \frac{1}{2} w_z^3 \right) \, dx dy dz \\ &= - \frac{1}{2} \int_{\Omega} (|\nabla_h w|^2 + w_z^2) w_z \, dx dy dz \\ &= \frac{1}{2} \int_{\Omega} (|\nabla_h w|^2 + w_z^2) (\nabla_h \cdot v) \, dx dy dz \\ &= - \frac{1}{2} \int_{\Omega} \nabla_h (|\nabla_h w|^2 + w_z^2) \cdot v \, dx dy dz. \end{aligned}$$

Then, from the above, we get

$$\begin{aligned} & \frac{1}{2} \frac{d (\|\nabla_h v\|_2^2 + \|v_z\|_2^2 + \|\nabla_h w\|_2^2 + \|w_z\|_2^2)}{dt} + \nu (\|\Delta_h v\|_2^2 + 2\|\nabla_h v_z\|_2^2 + \|v_{zz}\|_2^2 + \|\Delta_h w\|_2^2 + 2\|\nabla_h w_z\|_2^2 + \|w_{zz}\|_2^2) \\ &= \int_{\Omega} [(v \cdot \nabla_h) v - f] \cdot (\Delta_h v + v_{zz}) \, dx dy dz + \int_{\Omega} (\nabla_h w_z \cdot \nabla_h v + \nabla_h w \cdot \nabla_h v_z + w_{zz} v_z + w_z v_{zz}) \cdot v \, dx dy dz \\ &+ \int_{\Omega} [v \cdot \nabla_h w - g] (\Delta_h w + w_{zz}) \, dx dy dz - \frac{1}{2} \int_{\Omega} \nabla_h (|\nabla_h w|^2 + w_z^2) \cdot v \, dx dy dz. \end{aligned}$$

By applying Lemma 2 with some small enough  $\epsilon$  and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \frac{d(\|\nabla_h v\|_2^2 + \|v_z\|_2^2 + \|\nabla_h w\|_2^2 + \|w_z\|_2^2)}{dt} + \nu(\|\Delta_h v\|_2^2 + 2\|\nabla_h v_z\|_2^2 + \|v_{zz}\|_2^2 + \|\Delta_h w\|_2^2 + 2\|\nabla_h w_z\|_2^2 + \|w_{zz}\|_2^2) \\ & \leq C \left(1 + \|\tilde{v}\|_r^{\frac{2r}{r-3}} + \|\bar{v}\|_2^4 + \|\nabla_h \bar{v}\|_2^4\right) (\|\nabla_h v\|_2^2 + \|v_z\|_2^2 + \|\nabla_h w\|_2^2 + \|w_z\|_2^2) + \|f\|_2^2 + \|g\|_2^2. \end{aligned}$$

Thanks to Gronwall inequality, we obtain

$$\begin{aligned} & \|\nabla_h v(t)\|_2^2 + \|v_z(t)\|_2^2 + \|\nabla_h w(t)\|_2^2 + \|w_z(t)\|_2^2 \\ & + \nu \int_0^t (\|\Delta_h v(s)\|_2^2 + 2\|\nabla_h v_z(s)\|_2^2 + \|v_{zz}(s)\|_2^2 + \|\Delta_h w(s)\|_2^2 + 2\|\nabla_h w_z(s)\|_2^2 + \|w_{zz}(s)\|_2^2) ds \leq K_2, \end{aligned}$$

where

$$K_2 = e^{CT+K_{11}(T+K_{12}(T))+K_R^{2/(r-3)}T} [\|v_0\|_{H^1(\Omega)} + \|w_0\|_{H^1(\Omega)} + \|f\|_2^2 + \|g\|_2^2]. \quad (50)$$

Therefore, the  $H^1$  norm of the solution remains bounded on the maximal interval of existence  $[0, T^*)$ . This completes the proof of theorem.  $\square$

#### ACKNOWLEDGEMENTS

E.S.T. would like to thank the Bernoulli Center of the École Polytechnique Fédéral de Lausanne where part of this work was completed. This work was supported in part by the NSF grant No. DMS-0504619, the ISF grant No. 120/60 and the BSF grant No. 2004271.

#### REFERENCES

- [1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] L.C. Berselli, *On a regularity criterion for the solutions to the 3D Navier–Stokes equations*, Differential Integral Equations, **15** (2002), 1129–1137.
- [3] L.C. Berselli and G.P. Galdi, *Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations*, Proc. Amer. Math. Soc., **130** (2002), 3585–3595.
- [4] V.G. Bondarevsky, *On the problem of global regularity for 3-dimensional Navier–Stokes equations*, C. R. Math. Rep. Acad. Sci. Canada, **17** (1995), 109–114.
- [5] V.G. Bondarevsky, *A method of finding large sets of data generating global solutions to nonlinear equations: application to the Navier–Stokes equations*, C. R. Acad. Sci. Paris Sr. I Math., **322** (1996), 333–338.
- [6] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier–Stokes equations*, Handbook of Mathematical Fluid Dynamics. Vol. III, 161–244, North-Holland, Amsterdam, 2004.
- [7] C. Cao, J. Qin and E.S. Titi, *Regularity criterion of solutions for three-dimensional turbulent Channel flows*, arXiv:math.AP/0609667 v1 24 Sep 2006.
- [8] C. Cao and E.S. Titi, *Global well-posedness and finite dimensional global Attractor for a 3 – D planetary geostrophic viscous model*, Comm. Pure Appl. Math., **56** (2003), 198–233.
- [9] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Annal. Math., (to appear) (2007). <http://arxiv.org/abs/math/0503028>, 2005.
- [10] D. Chae and J. Lee, *Regularity criterion in terms of pressure for the Navier–Stokes equations*, Nonlinear Anal., **46** (2001), 727–735.
- [11] P. Constantin, *A few results and open problems regarding incompressible fluids*, Notices Amer. Math. Soc., **42** (1995), 658–663.
- [12] P. Constantin and C. Foias, *Navier–Stokes Equations*, The University of Chicago Press, 1988.
- [13] H.B. Da Veiga, *A sufficient condition on the pressure for the regularity of weak solutions to the Navier–Stokes equations*, J. Math. Fluid Mech., **2** (2000), 99–106.
- [14] C. Doering and J. Gibbon, *Applied Analysis of the Navier–Stokes Equations*, Cambridge University Press, 1995.
- [15] H. Fujita and T. Kato, *On the Navier–Stokes initial value problem*, I. Arch. Rat. Mech. Anal., **3** (1964), 269–315.
- [16] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. I & II, Springer-Verlag, 1994.
- [17] Y. Giga, *Solutions for semilinear parabolic equations in  $L^p$  and regularity of weak solutions of the Navier–Stokes system*, J. Differential Equations, **62** (1986), 186–212.
- [18] Y. Giga and T. Miyakawa, *Solutions in  $L_r$  of the Navier–Stokes initial value problem*, Arch. Rational Mech. Anal., **89** (1985), 267–281.

- [19] C. He, *New sufficient conditions for regularity of solutions to the Navier–Stokes equations*, Adv. Math. Sci. Appl., **12** (2002), 535–548.
- [20] T. Kato, *Strong  $L^p$  solutions of the Navier–Stokes equation in  $R^m$ , with applications to weak solutions*, Math. Z., **187** (1984), 471–480.
- [21] I. Kukavica and M. Ziane, *One component regularity for the Navier–Stokes equation*, **19(2)** (2006), 453–469.
- [22] O.A. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, English translation, 2nd ed., 1969.
- [23] O.A. Ladyzhenskaya, *Unique solvability in the large of three dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry*, Seminar in Mathematics, V.A. Steklov Mathematical Institute, Leningrad, 7, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part 2, Edited by O. A. Ladyzhenskaya, 1970, 70–79.
- [24] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, 1985.
- [25] O.A. Ladyzhenskaya, *The sixth millennium problem: Navier–Stokes equations, existence and smoothness*, (Russian) Uspekhi Mat. Nauk, **58(2)** (2003), (350), 45–78; translation in Russian Math. Surveys, **58(2)** (2003), 251–286.
- [26] S. Leibovich, A. Mahalov and E.S. Titi, *Invariant helical subspaces for the Navier–Stokes equations*, Arch. Rational Mech. Anal., **112** (1990), 193–222.
- [27] P.G. Lemarié-Rieusset, *Recent Developments in the Navier–Stokes Problem*, Chapman & Hall; London, 2002.
- [28] J.L. Lions, *Quelques Méthodes De Résolution Des Problèmes Aux Limites Non Linéaires*, Dunod, Paris, 1969.
- [29] P.L. Lions, *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, Oxford University Press, 1996.
- [30] Y. Meyer, *Oscillating patterns in some nonlinear evolution equations. Mathematical foundation of turbulent viscous flows*, 101–187, Lecture Notes in Math., **1871**, Springer, Berlin, 2006.
- [31] I. Moise, *Regularity of Navier–Stokes equations*, Differential Integral Equations, **19** (2006), 31–50.
- [32] M. Pokorný, *On the result of He concerning the smoothness of solutions to the Navier–Stokes equations*, Electron. J. Differential Equations **11** (2003), 1–8.
- [33] G. Prodi, *Un teorema di unicità per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl., **48** (1959), 173–182.
- [34] G. Raugel and G.R. Sell, *Navier–Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc., **6** (1993), 503–568.
- [35] G. Seregin and V. Sverák, *Navier–Stokes equations with lower bounds on the pressure*, Arch. Ration. Mech. Anal., **163(1)** (2002), 65–86.
- [36] J. Serrin, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–191.
- [37] H. Sohr, *The Navier–Stokes Equations, An Elementary Functional Analytic Approach*, Birkhäuser Verlag, Basel, 2001.
- [38] H. Sohr, *A regularity class for the Navier–Stokes equations in Lorentz spaces*, J. Evol. Equ., **1** (2001), 441–467.
- [39] H. Sohr, *A generalization of Serrin’s regularity criterion for the Navier–Stokes equations*, Quad. Mat., **10** (2002), 321–347.
- [40] R. Temam, *Navier–Stokes Equations, Theory and Numerical Analysis*, North-Holland, 1984.
- [41] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd Edition, SIAM, 1995.
- [42] R. Temam, *Some developments on Navier–Stokes equations in the second half of the 20th century*, Development of Mathematics 1950–2000, Birkhauser, Basel, 2000, 1049–1106.
- [43] Y. Zhou, *A new regularity criterion for the Navier–Stokes equations in terms of the gradient of one velocity component*, Methods Appl. Anal., **9** (2002), 563–578.
- [44] Y. Zhou, *Regularity criteria in terms of pressure for the 3–D Navier–Stokes equations in a generic domain*, Math. Ann., **328** (2004), 173–192.
- [45] Y. Zhou, *On regularity criteria in terms of pressure for the Navier–Stokes equations in  $\mathbb{R}^3$* , Proc. Amer. Math. Soc., **134** (2005), 149–156.

(C. Cao) DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, UNIVERSITY PARK, MIAMI, FL 33199, USA  
*E-mail address:* caoc@fiu.edu

(E.S. Titi) DEPARTMENT OF MATHEMATICS, AND DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92697-3875, USA, **also**, DEPARTMENT OF COMPUTER SCIENCE AND APPLIED MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL  
*E-mail address:* etiti@math.uci.edu and edriss.titi@weizmann.ac.il